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On the Question of Computation of the Dyadic Green's Function at the Source Region in Waveguides and Cavities

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Abstract—**The derivation of the dyadic Green's function for rectangular waveguides and cavities is approached systematically by using the theory of distributions. It is shown that, in order to obtain a complete solution for the field distribution in the entire structure, one must add an additional term to the classical expansion which is valid only outside the source region.**

I. INTRODUCTION

In a recent paper, Collin [1] has discussed the question of incompleteness of the E and H modes in the source region of a waveguide, and has shown that an additional term must be added to the classical representation of the field expression in order to derive a complete solution that is valid both in the source and source-free region of the waveguide. Tai [2] has also noted the difficulties involved in the computation of the dyadic Green's function, and has presented a solution based upon the use of eigenvector functions M and N. Neither of the preceding methods shares the simplicity of the conventional techniques used in solving the source-free waveguide problems, nor gives a clear picture of removing the difficulties involved in computing the dyadic Green's function in the source region.

The purpose of this short paper is to develop a systematic and novel approach for determining the dyadic Green's function and, consequently, the field distribution in the entire region of rectangular waveguides and cavities. It is shown that if one carefully defines the derivatives in the distribution sense and applies the correct completeness property of the modes, it is then possible to construct the complete solution for the entire structure just by employing the scalar eigenfunctions of the Helmholtz equation.

II. COMPUTATION OF THE DYADIC GREEN'S FUNCTION G_e AND FIELD DISTRIBUTION IN WAVEGUIDES

It is well known that the electric field excited by an electric current source can be expressed in terms of the dyadic Green's function $\mathsf{G}_e(r \mid r')$ as

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$$E(r) = -j\omega\mu \int \mathsf{G}_e(r \mid r') \cdot J(r')\, dv'. \tag{1}$$

One may notice from the preceding equation that care must be exercised in deriving $\mathsf{G}_e(r \mid r')$ at the source region if one wants to perform the preceding integration correctly. It is our purpose to derive the correct G_e in this section. The geometry of the problem with dimensions a and b along x and y axes, respectively, is a perfectly conducting rectangular waveguide aligned along the z axis and excited by a finite-volume electric current source.

To attack the problem, we write the Maxwell's equation and necessary boundary conditions as

$$\left.\begin{aligned} \nabla \times H &= J + j\omega\epsilon E \\ \nabla \times E &= -j\omega\mu H \end{aligned}\right\}, \quad \text{in the waveguide} \tag{2a}$$

$$\hat{n} \times E = 0, \qquad \text{on the wall.} \tag{2b}$$

It is assumed that the waveguide is filled with a homogeneous and isotropic material. In order to obtain the unique solution, the field components must also satisfy the Sommerfeld radiation condition along the z axis. From (2) one easily concludes that

$$\nabla \cdot H = 0, \qquad \text{in the waveguide} \tag{3a}$$

$$\hat{n} \cdot H = 0, \qquad \text{on the wall.} \tag{3b}$$

It is also trivial to show that E and H fields can be separated as follows:

$$\nabla \times \nabla \times E - k^2 E = -j\omega\mu J \tag{4a}$$

$$\nabla \times \nabla \times H - k^2 H = \nabla \times J. \tag{4b}$$

In order to solve the preceding boundary-value problem, we introduce the electric-type dyadic Green's function $\mathsf{G}_e(r \mid r')$ and magnetic-type dyadic Green's function $\mathsf{G}_m(r \mid r')$ by the following definitions:

$$\nabla \times \nabla \times \mathsf{G}_e - k^2 \mathsf{G}_e = \mathsf{I}\delta(r - r') \tag{5a}$$

$$\hat{n} \times \mathsf{G}_e = 0, \qquad \text{on the wall} \tag{5b}$$

and

$$\nabla \times \nabla \times \mathsf{G}_m - k^2 \mathsf{G}_m = \nabla \times \mathsf{I}\delta(r - r') \tag{6a}$$

$$\left.\begin{aligned} \hat{n} \cdot \mathsf{G}_m &= 0 \\ \hat{n} \times \nabla \times \mathsf{G}_m &= 0 \end{aligned}\right\}, \qquad \text{on the wall} \tag{6b}$$

where I is a unit dyadic. Notice that G_e and G_m are also subjected to the Sommerfeld radiation condition as discussed by Tai [3].

The first equation of (2a) shows that the following relation holds between G_m and G_e:

$$k^2 \mathsf{G}_e = \nabla \times \mathsf{G}_m - \mathsf{I}\delta(r - r'). \tag{7}$$

Note that (1) can be derived by applying the Green's theorem between E and G_e, and by making use of the boundary and radiation conditions.

The central issue regarding the completeness of modal representation of the dyadic Green's function is that $\nabla \cdot \mathsf{G}_e \neq 0$ in the waveguide. That $\nabla \cdot \mathsf{G}_e \neq 0$ is evident if the divergence operator is applied to both sides of (5a) and by recalling that $\nabla \cdot \mathsf{I}\delta(r - r') \neq 0$. Since $\nabla \cdot \mathsf{G}_e$ is not zero in the entire waveguide, G_e cannot be constructed in terms of the superposition of E modes alone (see, for instance, Goubau [4]). An additional term is necessary to obtain the complete form of G_e. In what follows, G_e will be found by using (7) after the complete form of G_m is derived. It should be noted that in contrast to G_e, G_m can be expressed in terms of H modes only, since $\nabla \cdot \mathsf{G}_m = 0$ everywhere.

In the following, a procedure is presented for constructing G_m which is believed to be simpler than Tai's method. Tai [2] used eigenvector functions M and N, which are divergenceless vectors to expand $\nabla \times \mathsf{I}\delta(r - r')$, and then he constructed G_m from (6a). In our procedure, we use the fact that $\nabla \cdot \mathsf{G}_m = 0$ and $\nabla \times \nabla \times =$

$\nabla\nabla\cdot-\nabla^2$ to obtain the following alternative form for (6), namely

$$(\nabla^2+k^2)\mathbf{G}_m=-\nabla\times\mathbf{I}\delta(\mathbf{r}-\mathbf{r}') \tag{8a}$$

$$\left.\begin{aligned}\hat{n}\cdot\mathbf{G}_m&=0\\ \hat{n}\times\nabla\times\mathbf{G}_m&=0\end{aligned}\right\},\quad\text{on the wall.} \tag{8b}$$

The preceding equation is an inhomogeneous Helmholtz equation which is readily solved by introducing another Green's function in the following manner:

$$(\nabla^2+k^2)\mathbf{g}_m=-\mathbf{I}\delta(\mathbf{r}-\mathbf{r}') \tag{9a}$$

$$\left.\begin{aligned}\hat{n}\cdot\mathbf{g}_m&=0\\ \hat{n}\times\nabla\times\mathbf{g}_m&=0\end{aligned}\right\},\quad\text{on the wall.} \tag{9b}$$

Applying Green's theorem, one obtains

$$\mathbf{G}_m(\mathbf{r}\mid\mathbf{r}')=\int\mathbf{g}_m(\mathbf{r}\mid\mathbf{r}'')\cdot\nabla''\times\mathbf{I}\delta(\mathbf{r}'-\mathbf{r}'')\,dv'' \tag{10}$$

where ∇'' means differentiation with respect to r'' coordinates. Simplification results through the introduction of $\mathbf{g}_m$ because it is a diagonalized dyadic, and therefore its elements may be obtained by simply solving a scalar Helmholtz equation in the following form:

$$(\nabla^2+k^2)\begin{Bmatrix}g_m^{xx}\\ g_m^{yy}\\ g_m^{zz}\end{Bmatrix}=-\begin{Bmatrix}1\\1\\1\end{Bmatrix}\delta(\mathbf{r}-\mathbf{r}') \tag{11a}$$

$$\left.\begin{aligned}\hat{n}\cdot\mathbf{g}_m&=0\\ \hat{n}\times\nabla\times\mathbf{g}_m&=0\end{aligned}\right\},\quad\text{on the wall.} \tag{11b}$$

To obtain the unique solution, the radiation condition must also be fulfilled. The boundary value problem (11) may be easily solved to obtain the following result:

$$\begin{aligned}\mathbf{g}_m(\mathbf{r}\mid\mathbf{r}'')=&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\epsilon_{0n}\epsilon_{0m}}{2ab\Gamma_{nm}}\exp(-\Gamma_{nm}|z-z''|)\\ &\cdot\Bigg[\hat{x}\hat{x}\left(\sin\frac{n\pi x}{a}\sin\frac{n\pi x''}{a}\cos\frac{m\pi y}{b}\cos\frac{m\pi y''}{b}\right)\\ &+\hat{y}\hat{y}\left(\cos\frac{n\pi x}{a}\cos\frac{n\pi x''}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y''}{b}\right)\\ &+\hat{z}\hat{z}\left(\cos\frac{n\pi x}{a}\cos\frac{n\pi x''}{a}\cos\frac{m\pi y}{b}\cos\frac{m\pi y''}{b}\right)\Bigg]\end{aligned} \tag{12}$$

where ϵ is an indicator with the following definition:

$$\epsilon_{0m}=\begin{cases}1, & m=0\\ 2, & m\neq 0\end{cases} \tag{13}$$

$$\Gamma_{nm}^2=k^2-\left(\frac{n\pi}{a}\right)^2-\left(\frac{m\pi}{b}\right)^2\quad\text{and}\quad k=\omega(\mu\epsilon)^{1/2}.$$

We now substitute (12) into (10) and perform the integration to solve for $\mathbf{G}_m$. This integration can be done in a simple way if one incorporates some elementary theorems of the distribution theory [5], [6]. To do this, the following equations are used to simplify the process of integration in (10):

$$\begin{aligned}\nabla''\times\mathbf{I}\delta(\mathbf{r}'-\mathbf{r}'')=&\left[\left(\frac{\partial}{\partial z''}\delta\right)\hat{y}-\left(\frac{\partial}{\partial y''}\delta\right)\hat{z}\right]\hat{x}+\left[\left(\frac{\partial}{\partial x''}\delta\right)\hat{z}\right.\\ &\left.-\left(\frac{\partial}{\partial z''}\delta\right)\hat{x}\right]\hat{y}+\left[\left(\frac{\partial}{\partial y''}\delta\right)\hat{x}-\left(\frac{\partial}{\partial x''}\delta\right)\hat{y}\right]\hat{z}\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial}{\partial z''}\exp(-\Gamma_{nm}|z-z''|)=&\Gamma_{nm}[\theta(z-z'')-\theta(z''-z)]\\ &\cdot\exp(-\Gamma_{nm}|z-z''|)\end{aligned} \tag{15}$$

where

$$\theta(z)=\begin{cases}1, & z>0\\ 0, & z<0\end{cases}$$

and

$$\int\left[\frac{\partial}{\partial z''}\delta(z-z'')\right]f\,dv''=-\left.\frac{\partial f}{\partial z''}\right|_{z''=z}. \tag{16}$$

Having employed (14), (15), and (16) in (10), one can easily perform the integration and obtain the following result:

$$\begin{aligned}\mathbf{G}_m(\mathbf{r}\mid\mathbf{r}')=&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\epsilon_{0n}\epsilon_{0m}}{2ab\Gamma_{nm}}\Bigg\{\Gamma_{nm}[\theta(z-z')-\theta(z'-z)]\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\\ &\cdot\cos\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}\hat{x}\hat{y}+\frac{m\pi}{b}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\cos\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\hat{x}\hat{z}\\ &-\frac{n\pi}{a}\cos\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\hat{y}\hat{z}\\ &-\Gamma_{nm}[\theta(z-z')-\theta(z'-z)]\\ &\cdot\cos\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\hat{y}\hat{x}\\ &-\frac{m\pi}{b}\cos\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\cos\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\hat{z}\hat{x}\\ &+\frac{n\pi}{a}\sin\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\cos\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}\hat{z}\hat{y}\Bigg\}\\ &\cdot\exp(-\Gamma_{nm}|z-z'|).\end{aligned} \tag{17}$$

The Green's function $\mathbf{G}_e$ is now derived from (17) with the aid of (7). To this end we use the following expressions, obtained from the application of the distribution theory, to simplify the result:

$$\begin{aligned}&\frac{\partial}{\partial z}\{[\theta(z-z')-\theta(z'-z)]\exp(-\Gamma_{nm}|z-z'|)\}\\ &\qquad=[-\Gamma_{nm}+2\delta(z-z')]\exp(-\Gamma_{nm}|z-z'|)\end{aligned} \tag{18}$$

and

$$\begin{aligned}\delta(x-x')\delta(y-y')&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\epsilon_{0n}\epsilon_{0m}}{ab}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\cos\frac{m\pi y}{a}\cos\frac{m\pi y'}{b}\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\epsilon_{0n}\epsilon_{0m}}{ab}\cos\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}.\end{aligned} \tag{19}$$

The preceding equation is known as the "completeness relation." Having employed (18) and (19) in the process of determining $\mathbf{G}_e$ from (7), one finally obtains the following equation:

$$\mathbf{G}_e(\mathbf{r}\mid\mathbf{r}')=\mathbf{G}_{e0}-\frac{1}{k^2}\hat{z}\hat{z}\delta(\mathbf{r}-\mathbf{r}') \tag{20}$$

where

$$\mathbf{G}_{e0}(\mathbf{r}\mid\mathbf{r}')=\frac{1}{k^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\epsilon_{0n}\epsilon_{0m}}{2ab\Gamma_{nm}}\exp(-\Gamma_{nm}|z-z'|)\left\{\left[\left(\frac{m\pi}{b}\right)^2-\Gamma_{nm}^2\right]\right.$$

$$
\begin{aligned}
&\cdot \cos\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\,\hat{x}\hat{x} + \left[\left(\frac{n\pi}{a}\right)^2 - \Gamma_{nm}^2\right]\sin\frac{n\pi x}{a}\\
&\cdot \sin\frac{n\pi x'}{a}\cos\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}\,\hat{y}\hat{y} + \left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]\sin\frac{n\pi x}{a}\\
&\cdot \sin\frac{n\pi x'}{a}\cos\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}\,\hat{z}\hat{z} - \frac{n\pi}{a}\frac{m\pi}{b}\cos\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\\
&\cdot \sin\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}\,\hat{x}\hat{y} - \frac{m\pi}{b}\frac{n\pi}{a}\\
&\cdot \sin\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\cos\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\,\hat{y}\hat{x}\\
&- \frac{m\pi}{b}\left[\theta(z-z') - \theta(z'-z)\right]\\
&\cdot \sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\cos\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\,\hat{y}\hat{z}\\
&+ \frac{m\pi}{b}\left[\theta(z-z') - \theta(z'-z)\right]\\
&\cdot \sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}\,\hat{z}\hat{y}\\
&+ \frac{n\pi}{a}\left[\theta(z-z') - \theta(z'-z)\right]\\
&\cdot \sin\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\,\hat{z}\hat{x}\\
&- \frac{n\pi}{a}\left[\theta(z-z') - \theta(z'-z)\right]\\
&\cdot \left.\cos\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\,\hat{x}\hat{z}\right\}. \qquad (21)
\end{aligned}
$$

We have introduced dyadic $\mathbf{G}_{e0}$ in (20), in order to emphasize the existence of $-(1/k^2)\hat{z}\hat{z}\delta(\mathbf{r}-\mathbf{r}')$ in the solution of $\mathbf{G}_e$. It is easy to check that the following relations hold:

$$\nabla\cdot\mathbf{G}_e(\mathbf{r}\mid\mathbf{r}') = 0, \qquad \mathbf{r}\neq\mathbf{r}' \tag{22a}$$

and

$$\mathbf{G}_e(\mathbf{r}\mid\mathbf{r}') = [\mathbf{G}_e(\mathbf{r}'\mid\mathbf{r})]^T \tag{22b}$$

where T indicates the transpose operation. Equation (22b) may be interpreted as the result of the reciprocity theorem. The preceding results were compared with those obtained by Tai [2], and perfect agreement was observed. It is very important to mention that, had the derivatives not been defined in the sense of distribution [(14)–(18)], the term $-(1/k^2)\hat{z}\hat{z}\delta(\mathbf{r}-\mathbf{r}')$ would have been missed very easily in the computation of (20).

Substituting (20) into (1), the E field may be derived in the entire waveguide as follows:

$$\mathbf{E} = -j\omega\mu\int \mathbf{G}_e\cdot\mathbf{J}\,dv' = -j\omega\mu\int\mathbf{G}_{e0}\cdot\mathbf{J}\,dv' - \frac{1}{j\omega\epsilon}J_z(\mathbf{r})\hat{z}. \tag{23}$$

One can readily show that Gauss' law, i.e., $\nabla\cdot\mathbf{E} = \rho/\epsilon$, is now satisfied.

It is interesting to note that the average power radiated by the current source depends only on $\mathbf{G}_{e0}$, viz.

$$P = \tfrac{1}{2}\,\mathrm{Re}\int \mathbf{E}\cdot\mathbf{J}^*\,dv \tag{24}$$

or

$$P = \tfrac{1}{2}\,\mathrm{Re}\; -j\omega\mu\iint [\mathbf{G}_{e0}(\mathbf{r}\mid\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')]\cdot\mathbf{J}^*(\mathbf{r})\,dv'\,dv. \tag{25}$$

III. COMPUTATION OF THE DYADIC GREEN'S FUNCTION $\mathbf{G}_e$ IN CAVITIES

In this section, we will construct the electric-type dyadic Green's function $\mathbf{G}_e$ for a rectangular cavity by following the steps developed in Section II. The geometry of a perfectly conducting rectangular cavity is constructed by choosing a, b, and c as the cavity dimensions along the x, y, and z axes, respectively.

The major difference in computation of $\mathbf{G}_e$ for the cavity compared with that of the waveguide is the functional form of $\mathbf{g}_m$, which, for the cavity, is determined to be

$$
\begin{aligned}
&\mathbf{g}_m(\mathbf{r}\mid\mathbf{r}'')\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\epsilon_{0n}\epsilon_{0m}\epsilon_{0l}}{abc\left[k^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 - \left(\frac{l\pi}{c}\right)^2\right]}\\
&\cdot\left\{\hat{x}\hat{x}\left(\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\cos\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}\cos\frac{l\pi z}{c}\cos\frac{l\pi z'}{c}\right)\right.\\
&+\hat{y}\hat{y}\left(\cos\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\cos\frac{l\pi z}{c}\cos\frac{l\pi z'}{c}\right)\\
&\left.+\hat{z}\hat{z}\left(\cos\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\cos\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}\sin\frac{l\pi z}{c}\sin\frac{l\pi z'}{c}\right)\right\}.
\end{aligned}
\tag{26}
$$

Substituting (26) into (10) and using (7), we finally obtain the following equation for $\mathbf{G}_e$ which is valid in the entire cavity:

$$\mathbf{G}_e(\mathbf{r}\mid\mathbf{r}') = \mathbf{G}_{e0}(\mathbf{r}\mid\mathbf{r}') - \frac{1}{k^2}(\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z})\delta(\mathbf{r}-\mathbf{r}') \tag{27}$$

where

$$
\begin{aligned}
\mathbf{G}_{e0} &= \frac{1}{k^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{\epsilon_{0n}\epsilon_{0m}\epsilon_{0l}}{abc\left[k^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 - \left(\frac{l\pi}{c}\right)^2\right]}\\
&\cdot\left\{\left[\left(\frac{m\pi}{b}\right)^2 + \left(\frac{l\pi}{c}\right)^2\right]\cos\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\right.\\
&\cdot\sin\frac{l\pi z}{c}\sin\frac{l\pi z'}{c}\,\hat{x}\hat{x} + \left[\left(\frac{l\pi}{c}\right)^2 + \left(\frac{n\pi}{a}\right)^2\right]\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\\
&\cdot\cos\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}\sin\frac{l\pi z}{c}\sin\frac{l\pi z'}{c}\,\hat{y}\hat{y} + \left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]\\
&\cdot\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\cos\frac{l\pi z}{c}\cos\frac{l\pi z'}{c}\,\hat{z}\hat{z}\\
&-\frac{n\pi}{a}\frac{m\pi}{b}\cos\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}\sin\frac{l\pi z}{c}\sin\frac{l\pi z'}{c}\,\hat{x}\hat{y}\\
&-\frac{m\pi}{b}\frac{n\pi}{a}\sin\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\cos\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\sin\frac{l\pi z}{c}\sin\frac{l\pi z'}{c}\,\hat{y}\hat{x}\\
&-\frac{m\pi}{b}\frac{l\pi}{c}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\cos\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}\sin\frac{l\pi z}{c}\cos\frac{l\pi z'}{c}\,\hat{y}\hat{z}\\
&-\frac{l\pi}{c}\frac{m\pi}{b}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y}{b}\cos\frac{m\pi y'}{b}\cos\frac{l\pi z}{c}\sin\frac{l\pi z'}{c}\,\hat{z}\hat{y}
\end{aligned}
$$

$$- \frac{l\pi}{c}\frac{n\pi}{a} \sin\frac{n\pi x}{a} \cos\frac{n\pi x'}{a} \sin\frac{m\pi y}{b} \sin\frac{m\pi y'}{b} \cos\frac{l\pi z}{c} \sin\frac{l\pi z'}{c}\, \hat{z}\hat{x}$$

$$- \frac{n\pi}{a}\frac{l\pi}{c} \cos\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \sin\frac{m\pi y}{b} \sin\frac{m\pi y'}{b} \sin\frac{l\pi z}{c} \cos\frac{l\pi z'}{c}\, \hat{x}\hat{z} \Bigg\}. \quad (28)$$

The main difference between (27) and (20) is the appearance of the terms $-(\delta(r - r')/k^2)\mathbf{I}$ and $(\delta(r - r')/k^2)\hat{z}\hat{z}$ in them, respectively. This indicates that if the current source is transverse in the waveguide, we will not obtain any term like $-(1/j\omega\epsilon)J_z(r)\hat{z}$ in the computation of the E field, whereas the components $-(1/j\omega\epsilon)\cdot [J_x(r)\hat{x} + J_y(r)\hat{y}]$ will show up in the computation of the E field in the cavity. The aforementioned difference between the equations results because we used the functional $\exp(-\Gamma_{nm} |z - z'|)$ for the waveguide case. This function has special characteristics when one computes its derivatives [(15) and (18)]. In other words, the infinite nature of the waveguide comes into the picture when one computes $\mathbf{G}_e$ for the waveguide.

IV. CONCLUSION

It is possible to construct the complete dyadic Green's function by employing the scalar eigenfunction of the Helmholtz equation. Care must be exercised in defining the derivatives in the sense of distribution and in using the correct completeness relation in order to compute the correct dyadic Green's function. This leads to the computation of the E field in the entire structure as well. The procedure discussed in this short paper may also be used to determine the complete form of the dyadic Green's function for nonrectangular waveguides and cavities.

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Enhancement of Surface-Acoustic-Wave Piezoelectric Coupling in Three-Layer Substrates

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***Abstract*—Numerical results with respect to the piezoelectric coupling of a three-layer substrate (CdS–SiO_2–Si) are presented. ⟨111⟩-cut Si is used, and the direction of propagation is [11$\bar{2}$], the SiO_2 layer is amorphous and the CdS layer (hexagonal, 6 mm) has the c axis normal to the substrate. The choice of these materials is connected with the integration of acoustic surface-wave devices on silicon. The interesting result is that, depending on the applied transducer configuration for particular values of kh of the SiO_2 and CdS layer, an increase in the piezoelectric coupling occurs. Its maximum exceeds the value of the coupling in the two-layer substrates: CdS–SiO_2 and CdS–Si, therefore more effective interdigital transducers can be designed for three-layer substrates.**

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The integration of frequency-selective acoustic surface-wave devices on silicon (Si) substrates has interesting device aspects [1]. Of particular interest are the structures which can be combined with p-n circuits and which can be realized by the application of silicon planar technology to one monolithic device. Since silicon is nonpiezoelectric, thin-film piezoelectric areas, including interdigital transducers, are necessary to convert electrical signal energy into acoustical energy. Cadmium sulfide (CdS) and zinc oxide (ZnO) are two materials commonly used in the construction of piezoelectric films.

To be compatible with silicon planar technology, the piezoelectric film must be separated from the silicon substrate by a (thermally grown) silicon dioxide (SiO_2) layer. This layer also provides an electrical insulation between the silicon substrate and the interdigital transducer, in case the configurations A or B are used (Fig. 1). In our calculations, however, the electrical conductivity of the silicon is assumed to be zero. This then leads to a substrate consisting of three layers, with Si and SiO_2 as essential materials. The ⟨111⟩-cut silicon wafer has a scribing flat in the ⟨11$\bar{2}$⟩ plane.

An important design parameter for surface-wave interdigital transducers is the piezoelectric coupling coefficient K that is a measure of electrical to acoustical energy conversion and vice versa. K has been calculated from the relative change $\Delta v/v$ in the phase velocity v, when an electrically perfect conducting plane is placed at the position of the interdigital transducer [2]; $K^2 = 2 \mid \Delta v/v \mid$ (approximately). The present calculations are an application of the theory of elastic wave propagation in thin layers, as given by Farnell and Adler [3], to a three-layer substrate CdS–SiO_2–Si. This is in contrast to Armstrong and Crampin [4], and Fahmy and Adler [5], who used a matrix formulation to perform these calculations on multilayer substrates.

The following specifications and assumptions are made (Fig. 1).

The layers are considered to be free of any loss.

The CdS layer (hexagonal, 6mm) has the c axis normal to the substrate surface.

The amorphous SiO_2 layer is considered electrically and elastically equivalent to fused quartz.

The Si substrate is ⟨111⟩ cut, coplanar with the X_1–X_2 plane and considered as a half-space.

The constants of the CdS and Si material were taken from Slobodnik and Conway [6] and those for used quartz were from Auld [7].

The mass loading due to the conducting planes is neglected.

The three-layer substrate is unbounded in the X_1–X_2 plane.

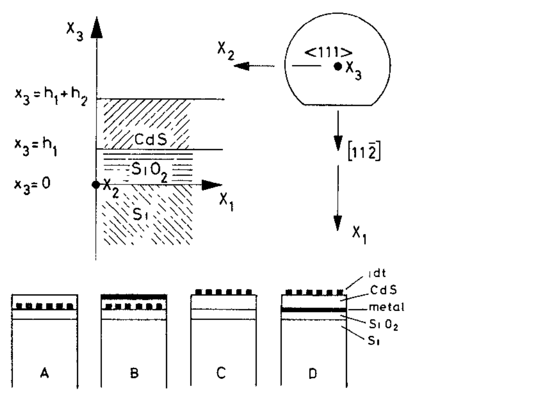

Fig. 1. Three-layer substrate with four possible transducer configurations.